Midterm 2: MAT 319

Instructions: Complete all problems below. You may not use calculators or other aides, including cell phones and books. Show all of your work. **Be sure to write your name and student ID on each page that you hand in.**

1.(25pts) Consider the function $f(x) = \sqrt{x}$ for $x \geq 0$. Provide a full proof of your answer to the following questions.

a) Is f uniformly continuous on $(0, 1]$?

b) Is f uniformly continuous on $[1, \infty)$?

a) Yes. Since f extends to a continuous function on $[0,1]$ by setting $f(0)=0$, by the theorem concerning equivalence of uniform continuity and such continuous extensions, f is uniformly continuous on $(0,1]$.

b) Yes. To see this observe that

$|f'(x)| = \frac{1}{2\sqrt{x}} \leq \frac{1}{2}$ for all $x \in [1, \infty)$.

Thus by the mean value theorem,

$|\sqrt{x} - \sqrt{y}| = |f'(c)||x-y| \leq \frac{1}{2}|x-y|$ for $x, y \in [1, \infty)$.

Then given $\varepsilon > 0$ we can choose $\delta = 2\varepsilon$ to obtain UC: $|f(x) - f(y)| < \varepsilon$, whenever $|x-y| < \delta$ and $x, y \in [1, \infty)$.

2.(25pts) Consider the sequence of functions $f_n(x) = \frac{1}{1+x^n}$ for $x \geq 0$ and $n = 1, 2, \ldots$. Provide a full proof of your answer to the following questions.

a) Compute the limit function $f(x) = \lim_{n\to\infty} f_n(x)$.

b) Does f_n converge uniformly to f on $[0,1]$?

a) $$f(x) = \begin{cases} 1 & 0 \leq x < 1 \\ 1/2 & x = 1 \\ 0 & x > 1 \end{cases}$$

This follows from the known limit:

$$\lim_{n\to\infty} x^n = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \\ \infty & x > 1 \end{cases}$$

b) No. If it converged uniformly on $[0,1]$ the limit function f would be continuous on $[0,1]$, since each f_n is continuous on this interval. However, f is clearly not continuous on $[0,1]$, yielding a contradiction.

3.(25pts) Show that $\sum_{n=1}^{\infty} \frac{1}{n^3} \sin(nx)$ converges uniformly on $\mathbb{R}$ to a continuous function.

Observe that $\frac{|\sin(nx)|}{n^3} \leq \frac{1}{n^3} =: M_n$ for all $x \in \mathbb{R}$, and $\sum_{n=1}^{\infty} M_n < \infty$ by the integral test. Thus, the Weierstrass M-test implies that the series converges uniformly on $\mathbb{R}$. Since each of the partial sums is continuous, the limit is also continuous due to the uniform convergence theorem.

4.(25pts) Let a function f be defined on $\mathbb{R}$, and assume that $|f(x)-f(y)| \le (x-y)^2$ for all $x, y \in \mathbb{R}$. Prove that f is a constant function.

We have that for any $y \in \mathbb{R}$,

$$|f'(y)| = \left| \lim_{x \to y} \frac{f(x)-f(y)}{x-y} \right|$$

$$= \lim_{x \to y} \frac{|f(x)-f(y)|}{|x-y|}$$

$$\le \lim_{x \to y} \frac{|x-y|^2}{|x-y|} = 0.$$

Hence $f = \text{const}$.